Quantum Time-Space Tradeoffs for Exponential Dynamic Programming

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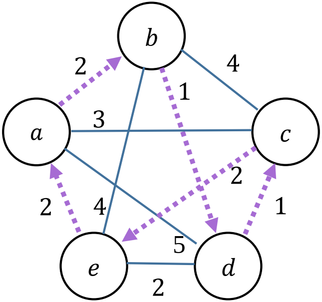
Travelling Salesman Problem

- Given a weighted graph on n vertices, find the shortest cycle that passes through each vertex exactly once.



A Question (Theory Days 2018)

- TRAVELLING SALESMAN PROBLEM:

Classically: [Bel62, HK62] $\widetilde{O}(2^n)$

Quantumly: [ABI+19] $\widetilde{O}(1.728^n)$

- *"Can the classical time-space tradeoffs of Koivisto and Parviainen be applied here? [KP10]"*

— Someone in the audience.

Time-Space Tradeoffs

- Less space ☺, more time ☹.
- Why? Let $D_1, \ldots, D_N$ be N memory records.
- Quantum Random Access Memory (QRAM):

$$\sum_{i=1}^{N} \alpha_i \left|i\right\rangle_{\text{address}} \left|0\right\rangle_{\text{target}} \longmapsto \sum_{i=1}^{N} \alpha_i \left|i\right\rangle_{\text{address}} \left|D_i\right\rangle_{\text{target}}$$

- $\mathrm{O}(\operatorname{polylog} N)$ in theory [GLM08], very hard to implement! [JR25]

[KP10] Time-Space Tradeoffs

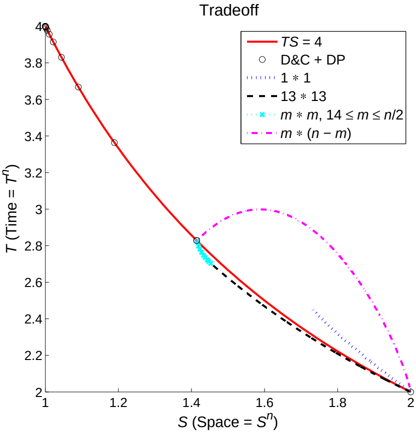


Figure 1: Space–time tradeoff schemes for permutation problems. The time requirement $O^*(T^n)$ is shown as a function of the space requirement $O^*(S^n)$, for $1 \leq S \leq 2$.

Roadmap

1. Classical Tradeoff $\longrightarrow$ Quantum Algorithm.

Gurevich and Shelah Tradeoff

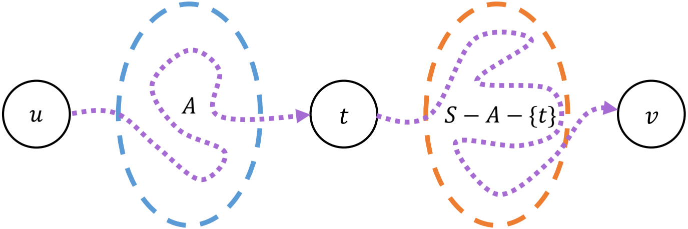
- TSP Algorithm 1: Divide & Conquer. Let $G = (V, E)$.
- $f(a, S, b)$ — the shortest path from a to b, visiting all of S.
- Optimal route:

$$\mathrm{OPT} = \min_{\substack{e \in V \\ \{1,e\} \in E}} \Big(f(1, V - \{1\} - \{e\}, e) + w(1, e) \Big).$$

Gurevich and Shelah Tradeoff

- D&Q recurrence:

$$f(u,S,v) = \min_{\substack{A\subset S\\ |A|=|S|/2\\ t\in S-A}} \Big(f(u,A,t) + f(t,S-A-\{t\},v)\Big).$$



Gurevich and Shelah Tradeoff

- TSP Algorithm 1: Divide & Conquer. Let $G = (V, E)$.
- Time complexity:

$$\binom{n}{n/2}\binom{n/2}{n/4}\cdots\binom{2}{1} = \mathrm{O}(2^n \cdot 2^{n/2} \cdots 2) = \mathrm{O}(4^n).$$

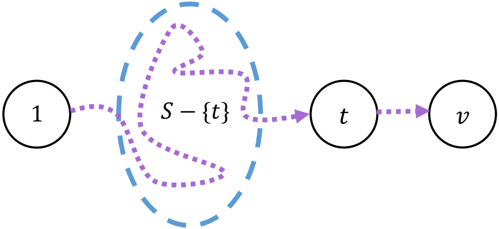
- Space complexity:

$$n + n/2 + \ldots + 1 = \mathrm{O}(n).$$

Gurevich and Shelah Tradeoff

- TSP Algorithm 2: Dynamic Programming.
- DP recurrence:

$$f(1, S, v) = \min_{t \in S} \Big(f(1, S - \{t\}, t) + w(t, v) \Big).$$



Gurevich and Shelah Tradeoff

- TSP Algorithm 2: Dynamic Programming.
- Time complexity:

$$|\{S \mid S \subseteq V\}| \cdot n^2 = \widetilde{\mathrm{O}}(2^n).$$

- Space complexity:

$$|\{S \mid S \subseteq V\}| \cdot n = \widetilde{\mathrm{O}}(2^n).$$

Gurevich and Shelah Tradeoff [GS87]

- TSP Tradeoff: pick a $k \in [\log_2 n]$.
  1. Divide & Conquer while $S > 2^k$.
  2. Dynamic Programming when $S = 2^k$.
- Time complexity: $\widetilde{\mathrm{O}}(4^n/2^k)$.
- Space complexity: $\widetilde{\mathrm{O}}(2^k)$.

Quantum Algorithm?

- Grover's search: for $x_1, \ldots, x_N$ can find

$$\min_{i=1}^{N} x_i$$

in $\widetilde{\mathrm{O}}(\sqrt{N})$ time and $\mathrm{O}(\log N)$ space.

Quantum Algorithm?

1. Dynamic Programming until sets of size αn for some $\alpha \in [0, 1/2]$.
2. Grover's search over the Divide & Conquer!

- Time complexity:

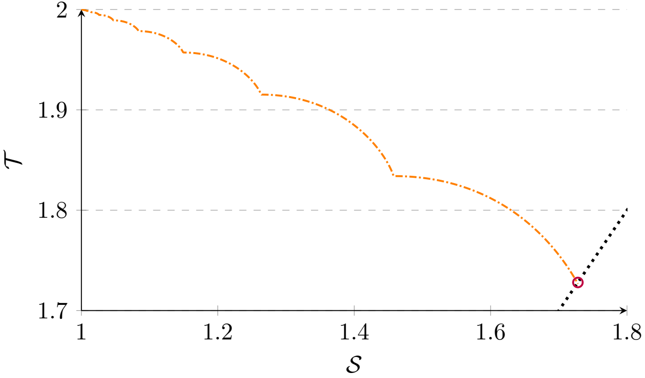
$$\widetilde{O}\left(\binom{n}{\alpha n} + \sqrt{\binom{n}{n/2}\binom{n/2}{n/4}\cdots\binom{n/2^{\log(1/\alpha)}}{\alpha n}}\right).$$

- Space complexity (QRAM):

$$\widetilde{O}\left(\binom{n}{\alpha n}\right).$$

Quantum Tradeoff

- By optimizing α, we get $\mathcal{T} = \mathcal{S} = \widetilde{\mathrm{O}}(1.728^n)$ [ABI+19].
- Other α choices give a time-space tradeoff.



Quantum Tradeoff

- Is this the best we can do?

Quantum Tradeoff

- Let's use Gurevich and Shelah again!

Roadmap

1. Classical Tradeoff ⟶ Quantum Tradeoff.
2. Quantum Tradeoff + Classical Tradeoff (again) ⟶ (an improved) Quantum Tradeoff.

Improved Quantum Tradeoff

The algorithm: pick two parameters $\alpha \in [0, 1/2]$ and $\beta \in [0, 1/2]$.

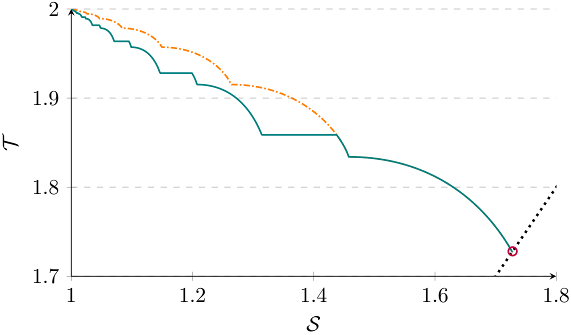
1. Run Grover's search + Divide & Conquer for sets larger than βn.
2. Else, use the [ABI+19] quantum algorithm with parameter α.

Improved Quantum Tradeoff

- By optimizing α and β, we obtain complexity

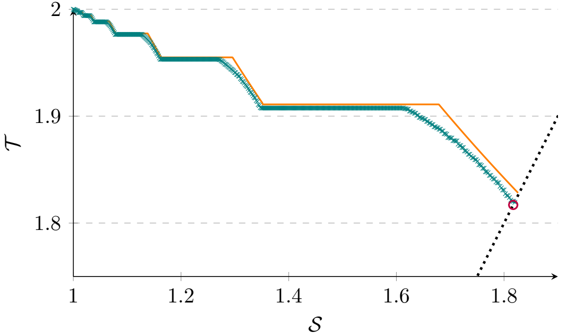
$$\mathcal{S} = \widetilde{\mathrm{O}}\left(2^{\mathrm{H}(\alpha)n}\right), \quad \mathcal{T} = \widetilde{\mathrm{O}}\left(2^{\max\left\{1-\frac{2-\mathrm{H}(2^k\alpha)}{2^{k+1}}, \mathrm{H}(\alpha)\right\}n}\right).$$

- An improved tradeoff:



Quantum Tradeoff for Permutation Problems

- We also optimize the permutation problem quantum algorithm of [ABI$^+$19], for PATHWIDTH, CUTWIDTH etc. (teal).
- We then "quantize" the *pairwise scheme* [PK09] time-space tradeoff (orange):



P.S. New Classical Improvements

- Koivisto and Parviainen: $(\mathcal{T}\mathcal{S})^{1/n} \leq 3.93$ [KP10].
- arxiv.org/abs/2604.05661: $(\mathcal{T}\mathcal{S})^{1/n} \leq 3.75$.
- arxiv.org/abs/2604.05645: $(\mathcal{T}\mathcal{S})^{1/n} \leq 3.58$.

Thank you!
Questions?

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