Coset Shaping for Coded Modulation

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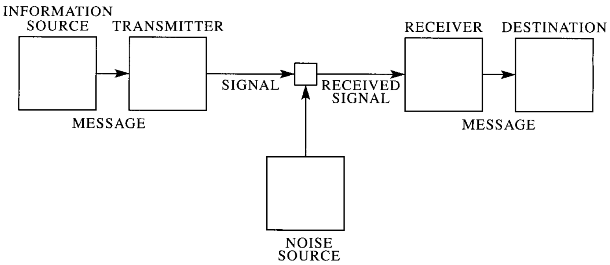
Theory of communication



Shannon, C.E., "A Mathematical Theory of Communication". Bell System Technical Journal, 27(3): 379-423, 1948.

Error-correcting codes

- **Linear code:** It is a k-dimensional linear subspace of $\mathbb{F}_q^n$, with basis $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_k$. Thus,

$$\mathscr{C} = \{\boldsymbol{c} \triangleq \boldsymbol{u}\mathsf{G},\ \text{for all } \boldsymbol{u} \in \mathbb{F}_q^k\},$$

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- **Rate:** Measures how much information is being transmitted, and is given by

$$\mathrm{R} = \frac{k}{n}.$$

- **Capacity:** Maximum possible rate at which information can be transmitted with error probability arbitrarily small.

Modulation

- The channel does not flip bits, it adds noise to signals

$$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{n},$$

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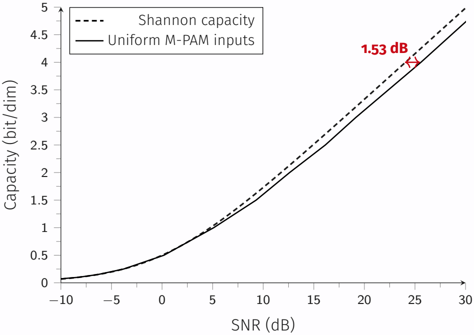
- **Modulation:** Transforms the bits (or coded bits) into physical signals

Table: BRBG code for 8-PAM

BRBG	000	001	011	010	110	111	101	100
8-PAM	-7	-5	-3	-1	1	3	5	7

Motivation

Conventional uniform QAM/PAM signaling has a gap of up to **1.53dB** to capacity in AWGN channels



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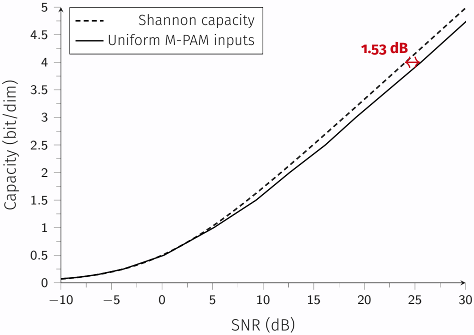
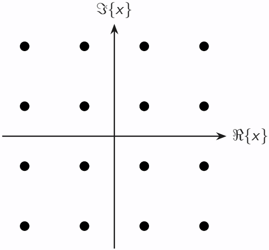


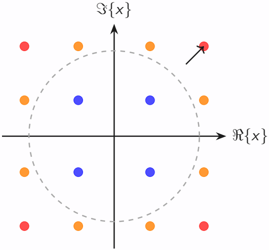
Goal: Reduce this gap via **shaping**!

Shaping

Geometric shaping

Equiprobable non-uniformly spaced signals.

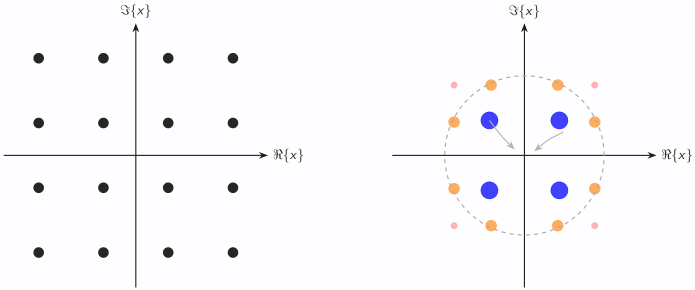




Shaping

Probabilistic shaping

Uniformly spaced signal points with different probabilities.



Shaping

Existing approaches:

- PAS (**P**robabilistic **A**mplitude **S**haping) shapes only information bits

G. Bocherer, ”Probabilistic parity shaping for linear codes,” arXiv:1902.10648, 2019.

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- PAS (**P**robabilistic **A**mplitude **S**haping) shapes only information bits
- Parity bit shaping raises complexity
- Voronoi shaping is good, but hard to combine with many codes

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Can we do better?

Idea

Coset shaping

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Coset shaping

New technique that shapes both <u>information</u> and <u>parity</u> bits

without increasing complexity

Coset shaping

- Use a linear error-correcting code of higher rate plus a shaping code defined by a set of coset leaders.
- Best coset leader chosen by the minimum energy criterion.

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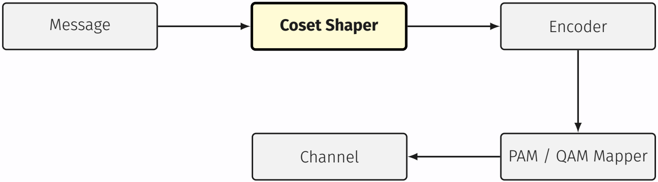
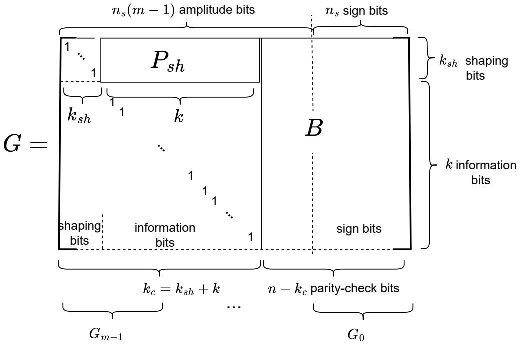


Coset shaping

- Integrates k_{sh} shaping bits into the generator matrix to define cosets



Encoding and decoding

Encoding

- For each message $\mathbf{u} \in \mathbb{F}_2^k$ to be sent:

1. Generate all $2^{k_{\text{sh}}}$ combinations of k_{sh} shaping bits $\mathbf{u}_{\text{sh}} \in \mathbb{F}_2^{k_{\text{sh}}}$
2. Compute the codeword candidates $\mathbf{v} = (\mathbf{u}_{\text{sh}}\ \mathbf{u})\text{G} \in \mathbb{F}_2^n$
3. Apply 2^m PAM modulation to each candidate $\mathbf{u}_{\text{sh}}$ and choose the one that minimizes the signal energy

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Decoding

- Standard decoding ignores shaping, then the coset leader is subtracted to recover the message bits.

Example

- Consider the $[6, 3]$-code generated by

$$G = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

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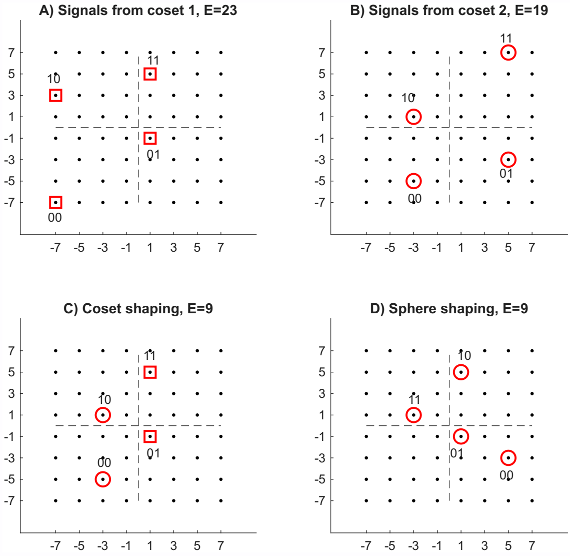
- The shaping-oriented form of the generator matrix is

$$G = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$



Y. C. Gültekin, T. Fehenberger, A. Alvarado, and F. M. Willems, "Probabilistic shaping for finite blocklengths: Distribution matching and sphere shaping," Entropy, vol. 22, no. 5, p. 581, 2020.

Asymptotics

Define an ensemble of coset codes as

$$\mathcal{C}_c = \{\mathscr{C}_c = \mathscr{C} \oplus \boldsymbol{a} : \mathscr{C} \in \mathcal{C}, \boldsymbol{a} \in \mathbb{F}_2^n\}$$

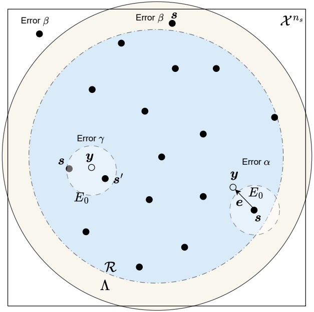
and the region

$$\Lambda = \Lambda(\mathcal{R}, E_0) = \{\boldsymbol{s} + \boldsymbol{e} : \boldsymbol{s} \in \mathcal{R}, \boldsymbol{e} \in E_0\},$$

where $E_0 = \mathrm{S}_{n_s}(\boldsymbol{x}, \sqrt{n_s(\sigma^2 + \delta)})$, $n_s = n/m$.

Decoding rule

If $\boldsymbol{y} = \boldsymbol{s}(\boldsymbol{u}) + E_0$ is received, then the decoder decides to $\boldsymbol{u}$. Otherwise, an error event is declared.



$$\mathrm{E}_\alpha = \boldsymbol{e} \notin E_0,$$

$$\mathrm{E}_\beta = \mathcal{R} \cap \left(\cup_{\boldsymbol{v}_{\mathrm{sh}} \in \mathscr{C}_{\mathrm{sh}}} \psi(\boldsymbol{v} \oplus \boldsymbol{v}_{\mathrm{sh}} \oplus \boldsymbol{a}) \right) = \varnothing$$

$$\mathrm{E}_\gamma = \boldsymbol{s}(\boldsymbol{u}') \in \boldsymbol{y} + E_0, \text{ for some } \boldsymbol{u}' \neq \boldsymbol{u}$$

Theorem

For large n, there exists a coset shaping construction achieving arbitrarily small error probability if the rate R satisfies:

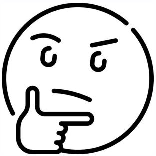
$$R \leq \frac{1}{2}\log_2\left(1 + {}^{P}/_{\sigma^2}\right) - \log_2\left(2\pi e\ G(\Lambda)\right) - o(n),$$

where $G(\Lambda)$ is the NSM of Λ, and $o(n) \to 0$ when $n \to \infty$.

Conjecture

If $n \to \infty$ then $G(\Lambda) \to 1/2\pi e$ and $R \to C$, where C denotes the AWGN capacity.

- How close is the shaping region to a sphere asymptotically?
- How linear is the nonlinear coset-shaped constellation?



Simulations

- Nonbinary QC-LDPC codes and 256-QAM
- Coset shaping outperforms PAS and sphere shaping
- Lower error floor and better energy efficiency

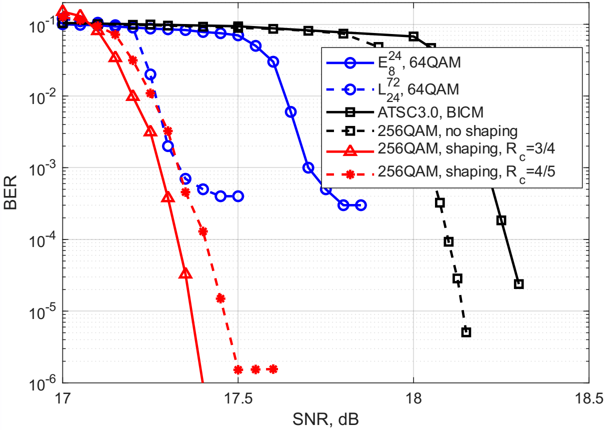


Figure: Comparison of NB LDPC coded QAM-256 signaling with and without shaping, $R_c = 5.33$ bits per QAM signal, Shannon limit is 15.97 dB, BICM limit is 17.02 dB, QAM limit is 15.99 dB

S. Li, A. Mirani, M. Karlsson and E. Agrell, "Coded Modulation Schemes for Voronoi Constellations," in IEEE Transactions on Communications, vol. 73, no. 10, pp. 8454-8469, Oct. 2025.

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- improves performance in practice and works for high-order constellations.

Thank You!