Monoidal categories graded by partial commutative monoids

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➜ This work defines an **algebraic structure** that has *as examples* various algebraic structures axiomatizing the **algebra of processes**, such as *monoidal categories* and *effectful categories*.

Informally, the kind of situation axiomatized by our structure is one having

- a collection of typed processes $X \xrightarrow{g} Y$, each with an associated *grade* (g), e.g.

bandwidth consumed	*memory locations accessed*
clearance level required	*whether the process has side-effects or not*

- sequential composites of the same grade $X \xrightarrow{g} Y \xrightarrow{g} Z$, and
- parallel composites, $\substack{X \xrightarrow{g} Y \\ U \xrightarrow[h]{} V}$, giving a process of *combined grade* $g \oplus h$, but *only sometimes*

Earnshaw, Nester, and Román, 2026 – *Monoidal categories graded by partial commutative monoids*

Parallel composition, subject to constraint

Consider a resource constrained by a numerical bound, such as *bandwidth* or *processor capacity* – e.g. processes may use some amount $r \in [0, 1]$ of the resource.

Then e.g. processes $X \xrightarrow{0.1} Y$ and $U \xrightarrow{0.8} V$ can be run in parallel

$$\begin{array}{c} X \xrightarrow{0.1} Y \\ U \xrightarrow[0.8]{} V \end{array}$$

resulting in a process that uses 0.9 of the resource.

But the parallel composite of the following two processes should not be defined

$$\begin{array}{c} U \xrightarrow{0.8} V \\ W \xrightarrow[0.3]{} Z \end{array}$$

Parallel composition, subject to constraint

Consider processes which might write to some memory locations $\{..., \ell_i, ...\} \subseteq \mathcal{L}$.

Then e.g. processes $X \xrightarrow{\{\ell_1,\ell_2\}} Y$ and $U \xrightarrow{\{\ell_3\}} V$ can be run in parallel

$$\begin{array}{c} X \xrightarrow{\{\ell_1,\ell_2\}} Y \\ U \xrightarrow[\{\ell_3\}]{} V \end{array}$$

since the memory locations accessed by each process are disjoint.

But the parallel composite of the following two processes should not be defined

$$\begin{array}{c} P \xrightarrow{\{\ell_1,\ell_2\}} Q \\ W \xrightarrow[\{\ell_1,\ell_3\}]{} Z \end{array}$$

Parallel composition, subject to constraint

Consider a programming language in which functions may either be *pure* or have *side-effects* (such as printing to the terminal).

Then programs $X \xrightarrow{\text{pure}} Y$ and $U \xrightarrow{\text{effectful}} V$ can be run in parallel

$$\begin{array}{c} X \xrightarrow{\text{pure}} Y \\ U \xrightarrow[\text{effectful}]{} V \end{array}$$

yielding an *effectful* program, and similarly for two *pure* programs.

But given another $W \xrightarrow{\text{effectful}} Z$,

$$\begin{array}{c} U \xrightarrow{\text{effectful}} V \\ W \xrightarrow[\text{effectful}]{} Z \end{array}$$

is not defined: effects might interfere.

Partial commutative monoids

Our *grades* are elements of *partial commutative monoids*.

Definition. A *partial commutative monoid* (PCM) $(E, \oplus, 0)$ is a set E, a partial function $\oplus : E \times E \rightharpoonup E$ and an element $0 \in E$ satisfying

$$a \oplus b \simeq b \oplus a,$$
$$a \oplus 0 = a = 0 \oplus a,$$
$$(a \oplus b) \oplus c \simeq a \oplus (b \oplus c),$$

where $\simeq$ means "if either side is defined, so is the other, and they are equal".

Definition. The *extension order* $(E, \leqslant)$ on the elements of a partial commutative monoid $(E, \oplus, 0)$ is the preorder defined by

$$a \leqslant b \text{ if and only if there exists } c \text{ such that } a \oplus c = b.$$

Examples of partial commutative monoids

- Commutative monoids, such as the trivial monoid **1**, are PCMs.
- The PCM **2** has two elements (0 and 1) with partial operation

$\oplus$	0	1
0	0	1
1	1	$\uparrow$

- The powerset of a set, $\mathscr{P}(X)$, is a PCM with operation

$$S \uplus T := \begin{cases} S \cup T & \text{if } S \cap T \text{ is empty} \\ \uparrow & \text{otherwise.} \end{cases}$$

- The real interval $[0, 1]$ is a PCM with the operation of bounded addition

$$x \dotplus y := \begin{cases} x + y & \text{if } x + y \leqslant 1 \\ \uparrow & \text{otherwise.} \end{cases}$$

PCM-graded monoidal categories

For a PCM $(E, \oplus, 0)$, an E-**graded monoidal category** consists of

- a monoid of objects, $(\mathbb{C}_{\text{obj}}, \otimes, I)$,
- for each grade, $a \in E$, a category $\mathbb{C}_a$ with set of objects $\mathbb{C}_{\text{obj}}$, with composition denoted by
$$(\fatsemi)_a : \mathbb{C}_a(X; Y) \times \mathbb{C}_a(Y; Z) \to \mathbb{C}_a(X; Z),$$
and identities at grade 0 denoted id_X,
- for each $a \leqslant b$ in the extension preorder of E, an identity-on-objects functor
$$(-)_a^b : \mathbb{C}_a \to \mathbb{C}_b,$$
- monoidal product operations for pairs $a, b \in E$ such that $a \oplus b$ is defined
$$(\otimes)_{a,b} : \mathbb{C}_a(X; Y) \times \mathbb{C}_b(X'; Y') \to \mathbb{C}_{a \oplus b}(X \otimes X'; Y \otimes Y').$$

PCM-graded monoidal categories (cont.)

This data is subject to the following axioms, whenever well typed.

(Reg-Act) $f^a_a = f$ and $(f^b_a)^c_b = f^c_a$, for $f \in \mathbb{C}_a$

(Reg-$\otimes$) $(f \otimes g)^{c \oplus d}_{a \oplus b} = f^c_a \otimes g^d_b$, for $f \in \mathbb{C}_a$, $g \in \mathbb{C}_b$,

($\otimes$-U-A) $f \otimes \mathrm{id}_I = f = \mathrm{id}_I \otimes f$ and $(f \otimes g) \otimes h = f \otimes (g \otimes h)$

($\otimes$-ID) $\mathrm{id}_X \otimes \mathrm{id}_Y = \mathrm{id}_{X \otimes Y}$

(Inter) $(f \otimes g) \fatsemi (h \otimes k) = (f \fatsemi h) \otimes (g \fatsemi k)$ whenever $a \oplus b$ is defined where $f \in \mathbb{C}_a(X; Y)$, $h \in \mathbb{C}_a(Y; Z)$, $g \in \mathbb{C}_b(X'; Y')$, and $k \in \mathbb{C}_b(Y'; Z')$.

Some basic properties of PCM-graded monoidal categories

Proposition. Let $f \in \mathbb{C}_a$ be a morphism in an E-graded monoidal category, where $a \leqslant b$. Then $f^b_a = f \otimes (\mathrm{id}_I)^c_0$, for every c witnessing $a \leqslant b$.

Proposition. Let $\mathbb{C}$ be an E-graded monoidal category. Each category $\mathbb{C}_e$ has a strict premonoidal structure

$$\begin{aligned} (A \ltimes -) &:= (\mathrm{id}_A \otimes_{0,e} -) : \mathbb{C}_e(X; Y) \to \mathbb{C}_e(A \otimes X; A \otimes Y) \\ (- \rtimes A) &:= (- \otimes_{e,0} \mathrm{id}_A) : \mathbb{C}_e(X; Y) \to \mathbb{C}_e(X \otimes A; Y \otimes A). \end{aligned}$$

Proposition. Let $\mathbb{C}$ be an E-graded monoidal category, and let $e = e \oplus e$ be an idempotent in E. Then $(\mathbb{C}_e, \otimes_{e,e}, I)$ is a strict monoidal category.

Effectful categories are $\mathbf{2}$-graded monoidal categories

Definition. A *morphism of PCM-graded monoidal categories* $(M, \phi) : (\mathbb{C}, E) \to (\mathbb{D}, F)$ comprises

- a monoid homomorphism $M : (\mathbb{C}_{\text{obj}}, \otimes_{\mathbb{C}}, I_{\mathbb{C}}) \to (\mathbb{D}_{\text{obj}}, \otimes_{\mathbb{D}}, I_{\mathbb{D}})$
- a PCM homomorphism $\phi : (E, \oplus_E, 0_E) \to (F, \oplus_F, 0_F)$, and
- functors $M_e : \mathbb{C}_e \to \mathbb{D}_{\phi(e)}$ with action M on objects,

satisfying two axioms (compat. of functors with regradings and $\otimes$).

Proposition. The category $\mathbf{2}$-GradMon is isomorphic to the category Eff of strict effectful categories and effectful functors.

Theorem. The full subcategory inclusion $i : \mathbf{2}\text{-GradMon} \hookrightarrow \text{GradMon}_{\top}$ has a right adjoint. That is, $\mathbf{2}$-GradMon is a coreflective subcategory of $\text{GradMon}_{\top}$.

Jeffrey, 1997 – *Premonoidal categories and a graphical view of programs*
Román, 2023 – *Promonads and string diagrams for effectful categories*

Freyd categories are cartesian **2**-graded monoidal categories

Definition. An E-graded monoidal category $\mathbb{C}$ is *symmetric* when $\mathbb{C}_0$ is symmetric strict monoidal and whenever $a \oplus b$ defined

$$(f \otimes g) \fatsemi (\sigma_{Y,Y'})_0^{a\oplus b} = (\sigma_{X,X'})_0^{a\oplus b} \fatsemi (g \otimes f).$$

An E-graded monoidal category is *cartesian* when $\mathbb{C}_0$ is cartesian monoidal, and the braiding there makes $\mathbb{C}$ into a symmetric E-graded monoidal category.

Proposition. The category of Freyd categories is isomorphic to the category of cartesian **2**-graded monoidal categories.

Proposition. Cartesian **3**-graded monoidal categories are the triples of Jeffrey.

Jeffrey, 1997 – *Premonoidal categories and a graphical view of programs*
Levy, Power, and Thielecke, 2003 – *Modelling environments in call-by-value programming languages*

PCM-graded monoidal categories are monoids

Proposition. Let $(E, \oplus, 0)$ be a partial commutative monoid. This induces a thin promonoidal structure $\mathbf{E} := ((E, \leqslant), P, I)$ on the extension preorder of E where

$$P(a, b; c) := \begin{cases} \top & \text{if } a \oplus b \leqslant c, \\ \varnothing & \text{otherwise,} \end{cases} \qquad I(c) := \top \text{ for all } c.$$

Theorem. An E-graded monoidal category with monoid of objects $(\mathbb{C}_{\mathrm{obj}}, \otimes, I)$ is precisely a monoid in the monoidal category

$$(\mathrm{MonCat}_{\mathrm{lax}}(\mathbb{C}_{\mathrm{obj}}^{\mathrm{op}} \times \mathbb{C}_{\mathrm{obj}}, ([\mathbf{E}, \mathrm{Set}], *, J)), \circ, L),$$

that is, a duoidally $[\mathbf{E}, \mathrm{Set}]$-enriched Freyd category.

Heunen and Sigal, 2023 – *Duoidally Enriched Freyd Categories*

Directions and questions

- (Partially) grading sequential composition in a compatible way?

$$\mathbb{C}_a(X; Y) \times \mathbb{C}_b(Y; Z) \to \mathbb{C}_{a \vee b}(X; Z)$$

- Partial commutative monoids widely used in separation logic – any substantial connections?
- Canonical refinements of effectful categories to more finely graded monoidal categories?