

# Monoidal categories graded by partial commutative monoids

Matt Earnshaw  
University of Tartu

joint work with Chad Nester (TÜ) & Mario Román (TTÜ)

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➔ This work defines an **algebraic structure** that has as *examples* various algebraic structures axiomatizing the **algebra of processes**, such as *monoidal categories* and *effectful categories*.

Informally, the kind of situation axiomatized by our structure is one having

- a collection of typed processes  $X \xrightarrow{g} Y$ , each with an associated *grade* ( $g$ ), e.g.

*bandwidth consumed*

*memory locations accessed*

*clearance level required*

*whether the process has side-effects or not*

- sequential composites of the same grade  $X \xrightarrow{g} Y \xrightarrow{g} Z$ , and

- parallel composites,  $\begin{matrix} X & \xrightarrow{g} & Y \\ U & \xrightarrow{h} & V \end{matrix}$ , giving a process of *combined grade*  $g \oplus h$ , but *only sometimes*

## Parallel composition, subject to constraint

Consider a resource constrained by a numerical bound, such as *bandwidth* or *processor capacity* – e.g. processes may use some amount  $r \in [0, 1]$  of the resource.

Then e.g. processes  $X \xrightarrow{0.1} Y$  and  $U \xrightarrow{0.8} V$  can be run in parallel

$$\begin{array}{c} X \xrightarrow{0.1} Y \\ U \xrightarrow{0.8} V \end{array}$$

resulting in a process that uses 0.9 of the resource.

But the parallel composite of the following two processes should not be defined

~~$$\begin{array}{c} U \xrightarrow{0.8} V \\ W \xrightarrow{0.3} Z \end{array}$$~~

## Parallel composition, subject to constraint

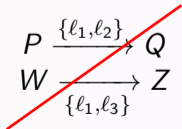
Consider processes which might write to some memory locations  $\{\dots, l_i, \dots\} \subseteq \mathcal{L}$ .

Then e.g. processes  $X \xrightarrow{\{l_1, l_2\}} Y$  and  $U \xrightarrow{\{l_3\}} V$  can be run in parallel

$$\begin{array}{c} X \xrightarrow{\{l_1, l_2\}} Y \\ U \xrightarrow{\{l_3\}} V \end{array}$$

since the memory locations accessed by each process are disjoint.

But the parallel composite of the following two processes should not be defined


$$\begin{array}{c} P \xrightarrow{\{l_1, l_2\}} Q \\ W \xrightarrow{\{l_1, l_3\}} Z \end{array}$$

## Parallel composition, subject to constraint

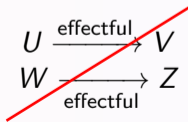
Consider a programming language in which functions may either be *pure* or have *side-effects* (such as printing to the terminal).

Then programs  $X \xrightarrow{\text{pure}} Y$  and  $U \xrightarrow{\text{effectful}} V$  can be run in parallel

$$\begin{array}{ccc} X & \xrightarrow{\text{pure}} & Y \\ U & \xrightarrow{\text{effectful}} & V \end{array}$$

yielding an *effectful* program, and similarly for two *pure* programs.

But given another  $W \xrightarrow{\text{effectful}} Z$ ,


$$\begin{array}{ccc} U & \xrightarrow{\text{effectful}} & V \\ W & \xrightarrow{\text{effectful}} & Z \end{array}$$

is not defined: effects might interfere.

## Partial commutative monoids

Our *grades* are elements of *partial commutative monoids*.

**Definition.** A *partial commutative monoid* (PCM)  $(E, \oplus, 0)$  is a set  $E$ , a partial function  $\oplus : E \times E \rightarrow E$  and an element  $0 \in E$  satisfying

$$\begin{aligned}a \oplus b &\simeq b \oplus a, \\a \oplus 0 &= a = 0 \oplus a, \\(a \oplus b) \oplus c &\simeq a \oplus (b \oplus c),\end{aligned}$$

where  $\simeq$  means “if either side is defined, so is the other, and they are equal”.

**Definition.** The *extension order*  $(E, \leq)$  on the elements of a partial commutative monoid  $(E, \oplus, 0)$  is the preorder defined by

$$a \leq b \text{ if and only if there exists } c \text{ such that } a \oplus c = b.$$

## Examples of partial commutative monoids

- Commutative monoids, such as the trivial monoid  $\mathbf{1}$ , are PCMs.
- The PCM  $\mathbf{2}$  has two elements (0 and 1) with partial operation

$\oplus$	0	1
0	0	1
1	1	$\uparrow$

- The powerset of a set,  $\mathcal{P}(X)$ , is a PCM with operation

$$S \uplus T := \begin{cases} S \cup T & \text{if } S \cap T \text{ is empty} \\ \uparrow & \text{otherwise.} \end{cases}$$

- The real interval  $[0, 1]$  is a PCM with the operation of bounded addition

$$x \dot{+} y := \begin{cases} x + y & \text{if } x + y \leq 1 \\ \uparrow & \text{otherwise.} \end{cases}$$

## PCM-graded monoidal categories

For a PCM  $(E, \oplus, 0)$ , an  $E$ -**graded monoidal category** consists of

- a monoid of objects,  $(\mathbb{C}_{\text{obj}}, \otimes, I)$ ,
- for each grade,  $a \in E$ , a category  $\mathbb{C}_a$  with set of objects  $\mathbb{C}_{\text{obj}}$ , with composition denoted by

$$(\circ)_a : \mathbb{C}_a(X; Y) \times \mathbb{C}_a(Y; Z) \rightarrow \mathbb{C}_a(X; Z),$$

and identities at grade 0 denoted  $\text{id}_X$ ,

- for each  $a \leq b$  in the extension preorder of  $E$ , an identity-on-objects functor

$$(-)_a^b : \mathbb{C}_a \rightarrow \mathbb{C}_b,$$

- monoidal product operations for pairs  $a, b \in E$  such that  $a \oplus b$  is defined

$$(\otimes)_{a,b} : \mathbb{C}_a(X; Y) \times \mathbb{C}_b(X'; Y') \rightarrow \mathbb{C}_{a \oplus b}(X \otimes X'; Y \otimes Y').$$

## PCM-graded monoidal categories (cont.)

This data is subject to the following axioms, whenever well typed.

$$(\text{REG-ACT}) \quad f_a^a = f \text{ and } (f_a^b)^c = f_a^c, \text{ for } f \in \mathbb{C}_a$$

$$(\text{REG-}\otimes) \quad (f \otimes g)_{a \oplus b}^{c \oplus d} = f_a^c \otimes g_b^d, \text{ for } f \in \mathbb{C}_a, g \in \mathbb{C}_b,$$

$$(\otimes\text{-U-A}) \quad f \otimes \text{id}_I = f = \text{id}_I \otimes f \text{ and } (f \otimes g) \otimes h = f \otimes (g \otimes h)$$

$$(\otimes\text{-ID}) \quad \text{id}_X \otimes \text{id}_Y = \text{id}_{X \otimes Y}$$

$$(\text{INTER}) \quad (f \otimes g) \circ (h \otimes k) = (f \circ h) \otimes (g \circ k) \text{ whenever } a \oplus b \text{ is defined where } f \in \mathbb{C}_a(X; Y), h \in \mathbb{C}_a(Y; Z), g \in \mathbb{C}_b(X'; Y'), \text{ and } k \in \mathbb{C}_b(Y'; Z').$$

## Some basic properties of PCM-graded monoidal categories

**Proposition.** Let  $f \in \mathbb{C}_a$  be a morphism in an  $E$ -graded monoidal category, where  $a \leq b$ . Then  $f_a^b = f \otimes (\text{id}_I)_0^c$ , for every  $c$  witnessing  $a \leq b$ .

**Proposition.** Let  $\mathbb{C}$  be an  $E$ -graded monoidal category. Each category  $\mathbb{C}_e$  has a strict premonoidal structure

$$\begin{aligned}(A \times -) &:= (\text{id}_A \otimes_{0,e} -) : \mathbb{C}_e(X; Y) \rightarrow \mathbb{C}_e(A \otimes X; A \otimes Y) \\ (- \times A) &:= (- \otimes_{e,0} \text{id}_A) : \mathbb{C}_e(X; Y) \rightarrow \mathbb{C}_e(X \otimes A; Y \otimes A).\end{aligned}$$

**Proposition.** Let  $\mathbb{C}$  be an  $E$ -graded monoidal category, and let  $e = e \oplus e$  be an idempotent in  $E$ . Then  $(\mathbb{C}_e, \otimes_{e,e}, I)$  is a strict monoidal category.

## Effectful categories are 2-graded monoidal categories

**Definition.** A morphism of PCM-graded monoidal categories  $(M, \phi) : (\mathbb{C}, E) \rightarrow (\mathbb{D}, F)$  comprises

- a monoid homomorphism  $M : (\mathbb{C}_{\text{obj}}, \otimes_{\mathbb{C}}, I_{\mathbb{C}}) \rightarrow (\mathbb{D}_{\text{obj}}, \otimes_{\mathbb{D}}, I_{\mathbb{D}})$
- a PCM homomorphism  $\phi : (E, \oplus_E, 0_E) \rightarrow (F, \oplus_F, 0_F)$ , and
- functors  $M_e : \mathbb{C}_e \rightarrow \mathbb{D}_{\phi(e)}$  with action  $M$  on objects, satisfying two axioms (compat. of functors with regradings and  $\otimes$ ).

**Proposition.** The category **2-GradMon** is isomorphic to the category **Eff** of strict effectful categories and effectful functors.

**Theorem.** The full subcategory inclusion  $i : \mathbf{2}\text{-GradMon} \hookrightarrow \text{GradMon}_{\top}$  has a right adjoint. That is, **2-GradMon** is a coreflective subcategory of  $\text{GradMon}_{\top}$ .

📖 Jeffrey, 1997 – *Premonoidal categories and a graphical view of programs*

📖 Román, 2023 – *Promonads and string diagrams for effectful categories*

## Freyd categories are cartesian 2-graded monoidal categories

**Definition.** An  $E$ -graded monoidal category  $\mathbb{C}$  is *symmetric* when  $\mathbb{C}_0$  is symmetric strict monoidal and whenever  $a \oplus b$  defined

$$(f \otimes g) \circ (\sigma_{Y, Y'})_0^{a \oplus b} = (\sigma_{X, X'})_0^{a \oplus b} \circ (g \otimes f).$$

An  $E$ -graded monoidal category is *cartesian* when  $\mathbb{C}_0$  is cartesian monoidal, and the braiding there makes  $\mathbb{C}$  into a symmetric  $E$ -graded monoidal category.

**Proposition.** The category of Freyd categories is isomorphic to the category of cartesian 2-graded monoidal categories.

**Proposition.** Cartesian 3-graded monoidal categories are the triples of Jeffrey.

📖 Jeffrey, 1997 – *Premonoidal categories and a graphical view of programs*

📖 Levy, Power, and Thielecke, 2003 – *Modelling environments in call-by-value programming languages*

## PCM-graded monoidal categories are monoids

**Proposition.** Let  $(E, \oplus, 0)$  be a partial commutative monoid. This induces a thin promonoidal structure  $\mathbf{E} := ((E, \leq), P, I)$  on the extension preorder of  $E$  where

$$P(a, b; c) := \begin{cases} \top & \text{if } a \oplus b \leq c, \\ \emptyset & \text{otherwise,} \end{cases} \quad I(c) := \top \text{ for all } c.$$

**Theorem.** An  $E$ -graded monoidal category with monoid of objects  $(\mathbb{C}_{\text{obj}}, \otimes, I)$  is precisely a monoid in the monoidal category

$$(\text{MonCat}_{\text{lax}}(\mathbb{C}_{\text{obj}}^{\text{op}} \times \mathbb{C}_{\text{obj}}, ([\mathbf{E}, \text{Set}], *, J)), \circ, L),$$

that is, a duoidally  $[\mathbf{E}, \text{Set}]$ -enriched Freyd category.

## Directions and questions

- (Partially) grading sequential composition in a compatible way?

$$\mathbb{C}_a(X; Y) \times \mathbb{C}_b(Y; Z) \rightarrow \mathbb{C}_{a \vee b}(X; Z)$$

- Partial commutative monoids widely used in separation logic – any substantial connections?
- Canonical refinements of effectful categories to more finely graded monoidal categories?