Equality of Proofs in Substructural Intuitionistic Modal Logics

Michel Smykalla
Co-work with Tarmo Uustalu, Niccolò Veltri & Cheng-Syuan Wan

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Let's start from the beginning...

Why intuitionistic modal logic?

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A large number of intuitionistic (constructive) modal logics (proof systems, semantics) have been defined and investigated. Their designs have been guided by different considerations:

- Semantical, proceeding from some mathematical (or possibly philosophical) flavor of semantics (Simpson, Ph.D. Thesis : The Proof theory and Semantics of I M. L.; Bierman and de Paiva: On an I.M.L.);

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- Combination of such considerations (Kavvos: Dual-Context Calculi for M.L.)

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- This work has been driven by our desire to understand this playground and to contribute our perspective, a “view from below”, starting from very weak logics defined by categories with very minimal structure.

But...

Why substructural?
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- The idea is that we should be able to add the structural rules of exchange, weakening and contraction separately if we want and that should not necessitate major reworking of the theory.
- The only connectives we consider are multiplicative truth (I), multiplicative conjunction ($\otimes$) and box ($\Box$).

And...

What about Proof Equality?

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The proof system we will is sound and complete regards to equality of proofs, wrt. the appropriate categorical semantics.

Outcomes

- We define non-commutative linear intuitionistic versions SIK, SIT, $\mathrm{SIK4}$, $\mathrm{SIS4}$ of modal logics K, T, K4 and S4 with truth, conjunction and box in terms of sequent calculi.

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- Each of these sequent calculi has cut admissible and is sound and complete, in particular in regards to equality of proofs, wrt. the appropriate categorical semantics. The box modality is interpreted as a strong monoidal functor in the case of SIK, and progressively strengthens to a strong monoidal comonad in the case of $\mathrm{SIS4}$

SIK system

Formulae are generated by the grammar

$$A, B ::= X \mid I \mid A \otimes B \mid \Box A$$

Derivations in SIK are generated by the rules: $\phi^i := \Box^i \phi$.

$$\frac{}{A^0 \vdash A}\ ax \qquad \frac{}{\vdash I}\ IR \qquad \frac{\Delta, \Gamma \vdash A}{\Delta, I^i, \Gamma \vdash A}\ IL^i \qquad \frac{\Delta \vdash A \quad \Gamma \vdash B}{\Delta, \Gamma \vdash A \otimes B}\ \otimes R$$

$$\frac{\Delta, A^i, B^i, \Gamma \vdash C}{\Delta, A \otimes B^i, \Gamma \vdash C}\ \otimes L^i$$

$$\frac{\Delta \vdash A}{\Delta^{1+} \vdash \Box A}\ \Box R \qquad \frac{\Delta, A^{i+1}, \Gamma \vdash B}{\Delta, \Box A^i, \Gamma \vdash B}\ \Box L^i$$

SIK system

SIK enjoys cut admissibility, though the cut rule is non-standard due to the presence of box-depth indices in the context.

Theorem

The following cut rule is admissible in SIK*:*

$$\frac{\Delta \vdash A \quad \Delta', A^i, \Gamma' \vdash C}{\Delta', \Delta^{i+}, \Gamma' \vdash C}\ cut$$

Equational Theory for SIK

We introduce an equivalence relation on proofs in SIK, which will be shown to axiomatize equality in the categorical models.

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A fragment of the equational theory for SIK

$$\dfrac{}{I^0 \vdash I}\,ax \quad \doteq \quad \dfrac{\dfrac{}{\vdash I}\,IR}{I^0 \vdash I}\,IL$$

$$\dfrac{}{\Box A^0 \vdash \Box A}\,ax \quad \doteq \quad \dfrac{\dfrac{\dfrac{}{A^0 \vdash A}\,ax}{A^1 \vdash \Box A}\,\Box R}{\Box A^0 \vdash \Box A}\,\Box L^1$$

$$\dfrac{\dfrac{\Delta, \Gamma \vdash A}{\Delta, I^i, \Gamma \vdash A}\,IL^i}{\Delta^{1+}, I^{1+i}, \Gamma^{1+} \vdash \Box A}\,\Box R \quad \doteq \quad \dfrac{\dfrac{\Delta, \Gamma \vdash A}{\Delta^{1+}, \Gamma^{1+} \vdash \Box A}\,\Box R}{\Delta^{1+}, I^{1+i}, \Gamma^{1+} \vdash \Box A}\,IL^{1+i}$$

$$\dfrac{\dfrac{\Delta, A^{i+1}, \Gamma \vdash B}{\Delta, \Box A^i, \Gamma \vdash B}\,\Box L^i}{\Delta^{1+}, \Box A^{1+i}, \Gamma^{1+} \vdash \Box B}\,\Box R \quad \doteq \quad \dfrac{\dfrac{\Delta, A^{i+1}, \Gamma \vdash B}{\Delta^{1+}, A^{1+i+1}, \Gamma^{1+} \vdash \Box B}\,\Box R}{\Delta^{1+}, \Box A^{1+i}, \Gamma^{1+} \vdash \Box B}\,\Box L^{1+i}$$

$$\dfrac{\dfrac{\Delta, A^{i+1}, \Gamma \vdash B \quad \Delta' \vdash C}{\Delta, A^{i+1}, \Gamma', \Delta' \vdash B \otimes C}\,\otimes R}{\Delta, \Box A^i, \Gamma, \Delta' \vdash B \otimes C}\,\Box L^i \quad \doteq \quad \dfrac{\dfrac{\Delta, A^{i+1}, \Gamma \vdash B}{\Delta, \Box A^i, \Gamma \vdash B}\,\Box L^i \quad \Delta' \vdash C}{\Delta, \Box A^i, \Gamma, \Delta' \vdash B \otimes C}\,\otimes R$$

$$\dfrac{\dfrac{\Delta, A^{i+1}, \Delta', \Gamma \vdash B}{\Delta, A^{i+1}, \Delta', I^j, \Gamma \vdash B}\,IL^j}{\Delta, \Box A^i, \Delta', I^j, \Gamma \vdash B}\,\Box L^i \quad \doteq \quad \dfrac{\dfrac{\Delta, A^{i+1}, \Delta', \Gamma \vdash B}{\Delta, \Box A^i, \Delta', \Gamma \vdash B}\,\Box L^i}{\Delta, \Box A^i, \Delta', I^j, \Gamma \vdash B}\,IL^j$$

Categorical Semantics for SIK

The sequent calculus SIK, together with the equality of proofs $\circeq$, also defines a monoidal category $(\mathcal{K}, \otimes, I)$ with a strong monoidal functor $\Box : \mathcal{K} \to \mathcal{K}$ validating the axioms $I \cong \Box I$ and $\Box A \otimes \Box B \cong \Box(A \otimes B)$. We call it the syntactic model. Objects of $\mathcal{K}$ are formulae and the homset $\mathcal{K}(A, B)$ is the set of proofs of sequent $A^0 \vdash B$ quotiented by $\circeq$. The identity morphism is the class of the axiom rule, while composition is defined using the admissible cut rule. The latter makes sense since cut preserves equality of proofs: if $f \circeq f'$ and $g \circeq g'$ then $cut(f, g) \circeq cut(f', g')$. The unit and the tensor are the connectives I and $\otimes$.

Categorical Completeness

Theorem

The syntactic model $\mathcal{K}$ is the free monoidal category with strong monoidal endofunctor on At.

Extensions: SIT, $\mathrm{SIK4}$ and $\mathrm{SIS4}$

System SIT is obtained by extending SIK with the following set of inference rules, one for each $i, j \in \mathbb{N}$:

$$\frac{\Delta, A^{i+j}, \Gamma \vdash B}{\Delta, A^{i+1+j}, \Gamma \vdash B}\ T^{i,j}$$

System $\mathrm{SIK4}$ is obtained by extending SIK with the following set of inference rules, one for each $i, j \in \mathbb{N}$:

$$\frac{\Delta, A^{i+2+j}, \Gamma \vdash B}{\Delta, A^{i+1+j}, \Gamma \vdash B}\ 4^{i,j}$$

Finally, system $\mathrm{SIS4}$ is obtained by extending SIK with the set of rules $T^{i,j}$ and $4^{i,j}$. In particular, all the extensions enjoy cut-admissibility.

Categorical Semantics for SIT, $\mathrm{SIK4}$ and $\mathrm{SIS4}$

Definition

A *strong monoidal copointed endofunctor* is a strong monoidal comonad without δ and the equations involving it. On the other hand, a *strong monoidal semicomonad* is a strong monoidal comonad without ϵ and the equations involving it. (ϵ is the counit and δ is the comultiplication).

The syntactic model of $\mathrm{SIS4}$ is constructed by extending the construction of $\mathcal{K}$, where the $\Box$ is a strong monoidal comonad. The corresponding models for SIT and $\mathrm{SIK4}$ are obtained by extending $\mathcal{K}$ where $\Box$ is copointed endofunctor and semicomonad, respectively.

Categorical Semantics for SIT, SIK4 and SIS4

Theorem

The syntactic model of SIS4 *(*SIT, SIK4*) is the free monoidal category with a strong monoidal comonad (copointed endofunctor, semicomonad) on* At.

Next Steps

- Extend the systems by linear implication;

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- Behind the correspondences of sequent calculi and categories with some structure as models, there is usually a correspondence between this relevant structure and an appropriate notion of multicategory. We plan to explore this in our situation.

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Thank you! 😄 😄 😄