Representability theory for multiactions

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Categories

A category $\mathcal{C}$ consists of:

▷ objects $a, b, c \ldots \in |\mathcal{C}|$

▷ arrows

$$\begin{tikzcd} a \arrow[d, "f"'] \\ b \end{tikzcd} \in \mathcal{C}(a,b)$$

IDENTITY ARROWS

$$\begin{tikzcd} a \arrow[d, "1_a"] \\ a \end{tikzcd}$$

COMPOSITION

$$\begin{tikzcd} a \arrow[d, "f"'] \arrow[dd, bend left, "g \circ f"] \\ b \arrow[d, "g"'] \\ c \end{tikzcd}$$

$\Rightarrow$ associative & unital

$$\underbrace{\begin{gathered}(h \circ g) \circ f = h \circ (g \circ f) \\ id_b \circ f = f = f \circ id_a\end{gathered}}$$

category $\equiv$
$\equiv$ „multi-sorted monoid"

one input, one output

How to formalise „multiple input" structures?

EXAMPLES

- $(X, \leq)$ … poset
  objects: $x \in X$
  arrows: $x \longrightarrow y$ iff $x \leq y$

- Mon
  objects: monoids
  arrows: monoid homomorphisms

Multiple inputs

A MONOIDAL CATEGORY is a category $\mathcal{C}$ together with:

- tensor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$

  $(a, b) \longmapsto a \otimes b$
- unit $i \in \mathcal{C}$
- associativity & unitality up to isomorphism

$$\alpha_{a,b,c} : (a \otimes b) \otimes c \xrightarrow{\cong} a \otimes (b \otimes c)$$

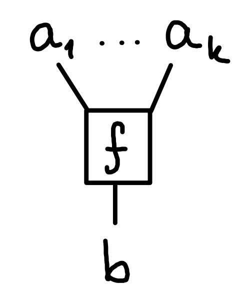
$$\rho_a : a \xrightarrow{\cong} a \otimes i$$

$$\lambda_a : a \xrightarrow{\cong} i \otimes a$$

+ 2 coherence axioms

A MULTICATEGORY consists of

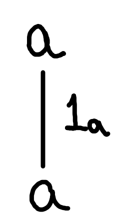
- objects $a, b, c \ldots \in |\mathbb{C}|$
- multiarrows



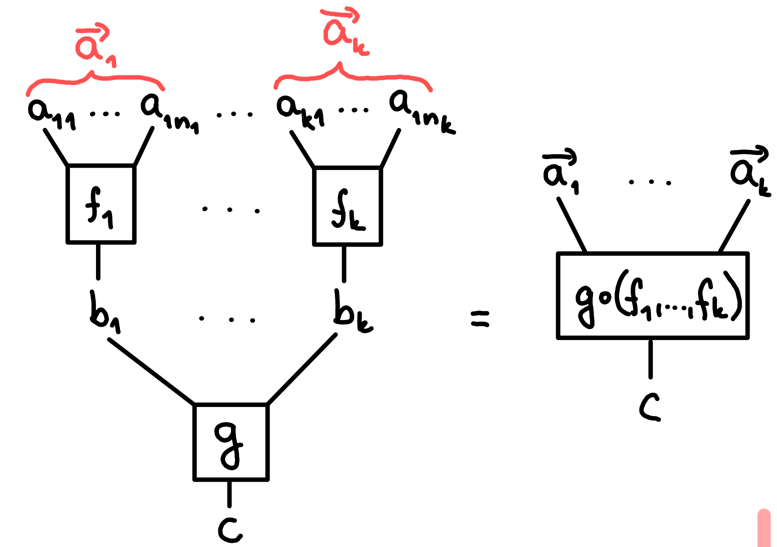
$\in \mathbb{C}(a_1, \ldots, a_k ; b)$

$k \in \mathbb{N}$

identity arrows



composition of multiarrows



associative & unital

EXAMPLES

Vector spaces over $\mathbb{k}$ with linear maps

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$$

$$\mathbb{k} \otimes V \cong V \cong V \otimes \mathbb{k}$$

Vector spaces over $\mathbb{k}$ with multilinear maps

$$V_1 \times \ldots \times V_k \longrightarrow W$$

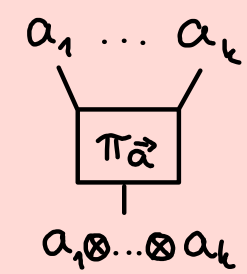
Representable multicategories

$\mathcal{M}$: (strict) monoidal category $\rightsquigarrow$ multicategory $\mathbb{M}$ with

$$\mathbb{M}(a_1,\dots,a_k;b) := \mathcal{M}(a_1\otimes\dots\otimes a_k;b)$$
$$\mathbb{M}(\,;b) := \mathcal{M}(i,b)$$

For the converse $\mathbb{M} \rightsquigarrow \mathcal{M}$ we need $\mathbb{M}$ REPRESENTABLE

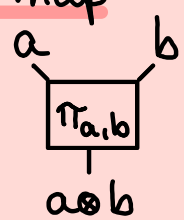
For every $a_1,\dots,a_k \in |\mathbb{M}|$ an object $a_1\otimes\dots\otimes a_k$ and (strong) universal in the sense that



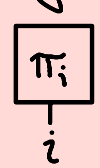
$$(-)\circ_i \pi_{\vec{a}} : \mathbb{M}(b_1,\dots,b_{i-1},a_1\otimes\dots\otimes a_k,b_{i+1},\dots,b_n;c) \cong$$
$$\cong \mathbb{M}(b_1,\dots,b_{i-1},a_1,\dots,a_k,b_{i+1},\dots,b_n;c)$$

Enough to specify (strong) universal

- binary map



- nullary map



INTUITION: $\mathbb{M}(a_1,a_2,a_3,\dots,a_{n-1},a_n;b)$

I can turn any of the commas , into tensors $\otimes$

C. Hermida (2000): Representable Multicategories

Theorem 9.8 There is a 2-equivalence

$$\mathrm{MonCat} \underset{}{\overset{\simeq}{\rightleftarrows}} \mathrm{Multicat}|_{\mathrm{rep}}.$$

Actions

An ACTION of a monoidal category $(\mathcal{M}, \otimes, i)$ consists of

- ▷ category $\mathcal{A}$
- ▷ $*: \mathcal{A} \times \mathcal{M} \longrightarrow \mathcal{A}$, $(a, m) \longmapsto a * m$
- ▷ associativity & unitality comparison maps

$$\alpha^{\mathcal{A}}_{a,m,n} : (a*m)*n \xrightarrow{\cong} a*(m \otimes n)$$

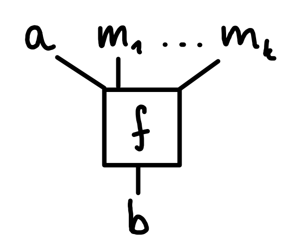
$$\rho^{\mathcal{A}}_{a} : a \xrightarrow{\cong} a \otimes i$$

\+ 3 coherence axioms

SKEW ACTION: $\alpha^{\mathcal{A}}, \rho^{\mathcal{A}}$ not iso

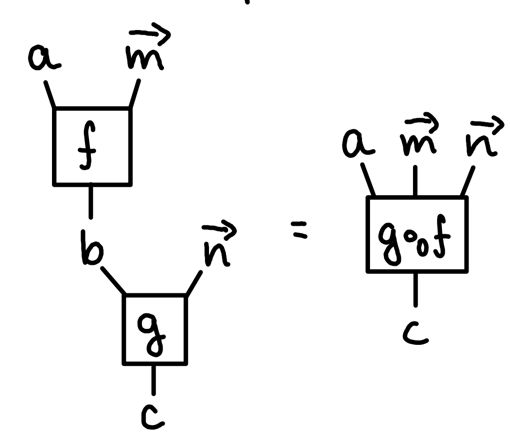
A MULTIACTION of a multicategory $\mathbb{M}$ consists of

- ▷ category $\mathbb{A}$
- ▷ for $a, b \in |\mathbb{A}|$, $m_1, \dots, m_k \in |\mathbb{M}|$ multiarrows

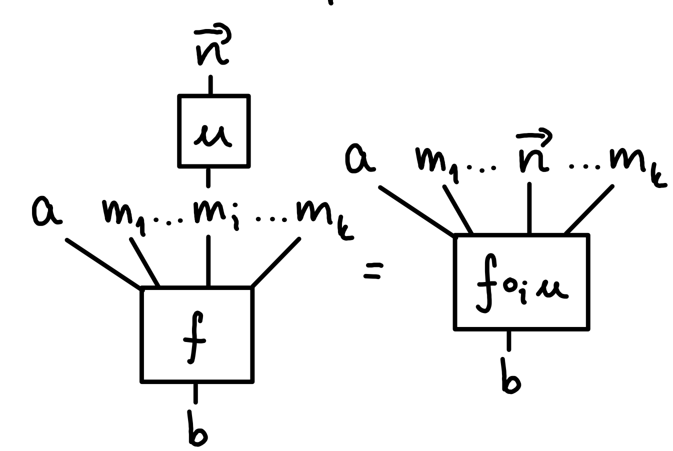


$\in \mathbb{A}(a, m_1, \dots, m_k; b)$

A-composition



M-composition



associative & unital

EXAMPLES

$\mathcal{C}$ … category

endofunctors: monoidal under $\circ$

$$\mathcal{C} \times [\mathcal{C}, \mathcal{C}] \xrightarrow{ev} \mathcal{C}$$
$$(a, F) \longmapsto Fa$$

$\mathcal{D} \subseteq [\mathcal{C}, \mathcal{C}]$ … subcategory (not monoidal)

$$\mathcal{C} \times \mathcal{D} \xrightarrow{ev} \mathcal{C}$$
$$(a, F) \longmapsto Fa$$

Representable multiactions

$\mathbb{A}$: $\mathbb{M}$-multiaction

$\mathbb{A}(a, m_1, \ldots, m_k; b) \qquad \mathbb{M}(m_1, \ldots, m_k; n)$

$\qquad\ \mathrm{I} \quad \mathrm{II} \qquad\qquad\qquad\qquad \mathrm{III}$

we have „three different types of commas" to turn into „tensors"

$\boxed{\mathrm{I}}$ $\quad (-) \circ_0 \boxed{\vartheta_{a,m}} \;:\; \mathbb{A}(a * m, m_1, \ldots, m_k; b) \cong \mathbb{A}(a, m, m_1, \ldots, m_n; b)$

$\boxed{\mathrm{II}}$ $\quad (-) \circ_i^{\mathbb{A}} \boxed{\pi_{m_i, m_{i+1}}} \;:\; \mathbb{A}(a, m_1, \ldots, m_i \otimes m_{i+1}, \ldots, m_k; b) \cong \mathbb{A}(a, m_1, \ldots, m_i, m_{i+1}, \ldots, m_k; b)$

$\boxed{\mathrm{III}}$ $\quad (-) \circ_i^{\mathbb{M}} \boxed{\pi_{m_i, m_{i+1}}} \;:\; \mathbb{M}(m_1, \ldots, m_i \otimes m_{i+1}, \ldots, m_k; n) \cong \mathbb{M}(m_1, \ldots, m_i, m_{i+1}, \ldots, m_k; n)$

Representable multiactions

$\mathcal{A}$: $\mathbb{M}$-multiaction $\rightsquigarrow$ $\mathcal{A}(a, m_1, \dots, m_k ; b)$ $\quad$ $\mathbb{M}(m_1, \dots, m_k ; n)$

$\boxed{\mathrm{I}}$ → $\mathcal{A}$ representable

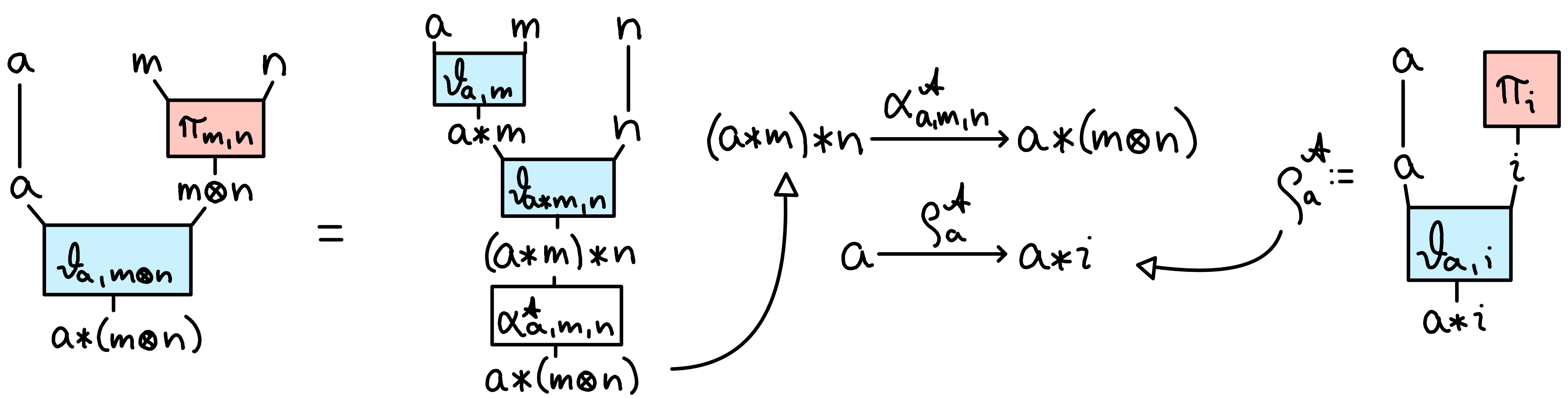
$\boxed{\mathrm{II}}$, $\boxed{\mathrm{III}}$ → $\mathcal{A}$ respects $\mathbb{M}$-representability

$\boxed{\mathrm{III}}$ → $\mathbb{M}$ representable

Prop: $\boxed{\mathrm{I}}$ $\quad$ $\{\mathcal{A}$ representable $\mathbb{M}$-multiact.$\} \simeq \{\mathcal{A}$ action of $F\mathbb{M}\}$

free strict mon. cat. on $|\mathbb{M}|$

Prop: $\boxed{\mathrm{I}} + \boxed{\mathrm{III}}$ $\quad$ $\left\{\begin{matrix}\mathcal{A} \text{ representable } \mathbb{M}\text{-multiact.} \\ \& \ \mathbb{M} \text{ representable}\end{matrix}\right\} \simeq \{\mathcal{A}$ skew action of $\mathcal{M}\}$



$$(a*m)*n \xrightarrow{\alpha^{\mathcal{A}}_{a,m,n}} a*(m\otimes n)$$

$$a \xrightarrow{\varrho^{\mathcal{A}}_a} a*i \qquad \varrho^{\mathcal{A}}_a :=$$

Prop: $\boxed{\mathrm{I}} + \boxed{\mathrm{II}} + \boxed{\mathrm{III}}$ $\quad$ $\left\{\begin{matrix}\mathcal{A} \text{ rep. } \mathbb{M}\text{-multiact}, \mathbb{M} \text{ rep.} \\ \& \ \mathcal{A} \text{ respects } \mathbb{M}\text{-rep.}\end{matrix}\right\} \simeq \{\mathcal{A}$ (strong) action of $\mathcal{M}\}$

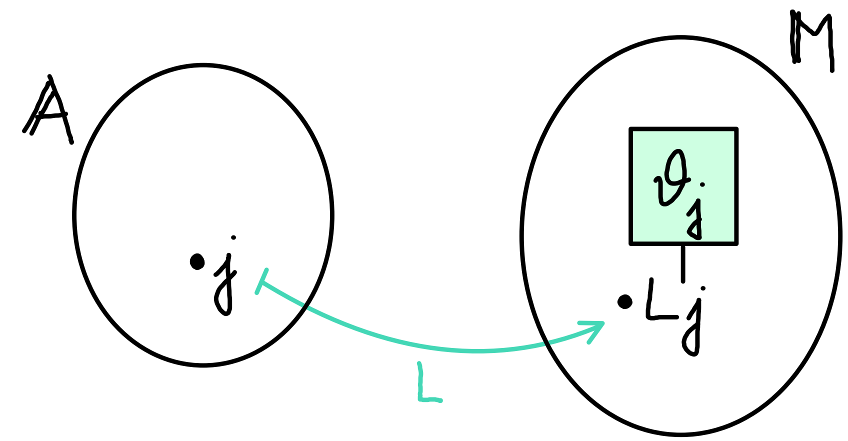
Skew monoidal categories

We can also consider morphisms of multiactions

Especially, we care about $\mathbb{A}$: $\mathbb{M}$-multiaction

$L : \mathbb{A} \longrightarrow \mathbb{M}$ $\quad$ $\mathbb{M}$-multiact. morphism

Now, we can also deal with unit objects in $\mathbb{A}$:



$$L(-)\circ\vartheta_j \quad \mathbb{A}(j, m_1, \dots, m_k; b) \cong \mathbb{M}(m_1, \dots, m_k; Lb)$$

This allows us to capture:

- skew monoidal categories
- skew warpings on actions
- skew action together with
$$\begin{tikzcd} \mathcal{A} \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, bend right, "j*(-)"'] \end{tikzcd}$$
- skew actions of skew monoidal categories

References

Hermida, C. (2000). Representable multicategories. Advances in Mathematics, 151(2), 164-225.

Some particular versions of representability of multiactions and their morphisms were treated in:

Bourke J., Lack, S. (2018). Skew monoidal categories and skew multicategories. Journal of algebra, 506, 237-266.

Stroiński, M., Zorman, T. (2024). Reconstruction of module categories in the infinite and non-rigid settings
arXiv: 2409.00793