Two approaches to convergence of trajectories

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Introduction

- A point x of a metric space (X, d) under the action of a continuous transformation $T : X \to X$ determines a *trajectory* as a function from $\mathbb{N}$ (or even from $\mathbb{Z}$ if T has a continuous inverse) to X: that is, a (possibly bi-)*infinite word*.
- The distance d determines a notion of convergence of a sequence of points.
- What can be a good notions of convergence of trajectories?
- We illustrate two approaches to this issue, due to Abram Besicovitch and Hermann Weyl, in the specific case where X is a finite alphabet with the *discrete distance*.
- We discuss an application to the theory of *cellular automata* in one dimension, and discuss what changes when more generic structures in space are considered.

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Terminology

- We assume all alphabets to have at least two elements.
- If $E(x)$ is an expression that, for every element x of X, describes some element of Y, we denote by $\lambda x : X \,.\, E(x) : Y$ the function that associates to every $x \in X$ the element of Y described by $E(x)$. We omit X and/or Y if irrelevant or clear from the context.
- We denote the collection of the finite subsets of a set X as $\mathcal{PF}(X)$.
- The *slice* of integers from m to n is $[m:n] = \{k \in \mathbb{Z} \mid m \leqslant k \leqslant n\}$.
- The *Hamming distance* of f and g relative to a finite subset U of X is the cardinality $H_U(f,g)$ of the set $\{x \in U \mid f(x) \neq g(x)\}$.
- The *Iverson brackets* on a commutative monoid $(A, +, 0)$ are the function $\cdot\,[\cdot]_I : A \to \{\mathtt{true}, \mathtt{false}\} \to A$ defined by:

$$a\,[P]_I = \text{if } P \text{ then } a \text{ else } 0$$

with the additional convention that if a is undefined, then $a\,[\mathtt{false}]_I = 0$.
If $(A, +, 0, \cdot, 1)$ is also a *semiring with identity*, we write $[P]_I$ as a shorthand for $1\,[P]_I$.

Dynamical systems

A *dynamical system* is a pair (X, T) where X is a (usually compact) metric space (with a distance d) and $T : X \to X$ is a continuous function.

- A *subsystem* of (X, T) is a closed set $U \subseteq X$ such that $T(U) \subseteq U$.
  A subsystem is *minimal* if it has no proper subsystem.
- The *(forward) trajectory* of the point $x \in X$ is the function $x^{(\cdot)} : \mathbb{N} \to X$ defined recursively as $x^{(0)} = x$ and $x^{(n+1)} = Tx^{(n)}$ for every $n \in \mathbb{N}$.
- If T has a continuous inverse, then we can define the trajectory as a function from $\mathbb{Z}$ to X by $x^{(-(n+1))} = T^{-1}x^{(-n)}$ for every $n \in \mathbb{N}$.
- The *orbit* of $x \in X$ is the set $\mathcal{O}(x)$ of the points $x^{(n)}$.

A basic example: The shift dynamical system

Let A^ω be the set of *infinite words* over a finite alphabet A.

- We denote the symbol at position i in u as u_i, and the finite word $u_m \dots u_n$ as $u_{[m:n]}$.
- The distance on A^ω defined by:

$$d(u, w) = \begin{cases} 2^{-n} & \text{if } u \neq w \text{ and } n = \min\{i \mid u_i \neq w_i\} , \\ 0 & \text{if } u = w \end{cases}$$

  induces the *prodiscrete topology* (product of countably many discrete copies of A) and makes A^ω a *Cantor space* (compact, perfect, totally disconnected, zero-dimensional).
- The *shift map* is the function $\sigma : \{0,1\}^\omega \to \{0,1\}^\omega$ defined by:

$$\sigma(u)_n = u_{n+1} \text{ for every } n \in \mathbb{N}$$

- The pair (A^ω, σ) is the *shift dynamical system* on the alphabet A.

A similar construction can be applied to *bi-infinite words*, with the distance:

$$d(u, w) = \begin{cases} 2^{-n} & \text{if } u \neq w \text{ and } n = \min\{|i| \mid u_i \neq w_i\} , \\ 0 & \text{if } u = w \end{cases}$$

A short list of dynamical properties

A dynamical system (X, T) is:

- *equicontinuous at x* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $d(x, y) < \delta$ it is $d(T^n x, T^n y) < \varepsilon$ for *every* n;
- *equicontinuous* if it is equicontinuous at every x;
- *transitive* if for every two nonempty open sets U, V there are $x \in U$ and n such that $T^n x \in V$;
- *sensitive (to initial conditions)* if there exists $\varepsilon_0 > 0$ with the following property: For every $x \neq y$ there exists n such that $d(T^n x, T^n y) \geqslant \varepsilon_0$;
- *chaotic* if it is transitive, sensitive to initial conditions, and the set of *periodic points* (such that $T^n x = x$ for some $n \neq 0$) is dense in X.

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The shift dynamical system is chaotic!

Symbolic dynamics

Hadamard, 1898:

- Inspiration from the theory of geodesic flows on surfaces of negative curvature.
- Given a finite partition of the surface, keep track of the *current block at a given time*.
- Then there is a *finite* set of “forbidden pairs” such that the possible sequences are those that do *not* contain any forbidden pair!

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Then what does it mean for two trajectories to be "near" or "far"?

The Besicovitch pseudodistance

Let X be a *countable* set, Y a *finite* set, and $\mathcal{X} = (X_n) \subseteq \mathcal{PF}(X)$ a nondecreasing sequence such that $\bigcup_{n \geqslant 0} X_n = X$.

- The *Besicovitch pseudodistance* associated to $\mathcal{X}$ is the function $d_{\mathcal{B},\mathcal{X}} : Y^X \times Y^X \to [0,1]$ defined by:

$$d_{\mathcal{B},\mathcal{X}}(f,g) = \limsup_{n \to \infty} \frac{1}{|X_n|} H_{X_n}(f,g) \text{ for every } f,g : X \to Y\,.$$

- The *Besicovitch equivalence* on Y^X associated to $\mathcal{X}$ is defined as:

$$f \sim_{\mathcal{B},\mathcal{X}} g \text{ if and only if } d_{\mathcal{B},\mathcal{X}}(f,g) = 0\,.$$

- The *Besicovitch space* associated to $\mathcal{X}$ is the metric space:

$$\mathcal{B}_{\mathcal{X}}(X,Y) = \left(Y^X / \sim_{\mathcal{B},\mathcal{X}}, d_{\mathcal{B},\mathcal{X}} \right)$$

  where $d_{\mathcal{B},\mathcal{X}}(\varphi,\gamma) = d_{\mathcal{B},\mathcal{X}}(f,g)$ for any $f \in \varphi$ and $g \in \gamma$.

We omit X, Y, and/or $\mathcal{X}$ if irrelevant or clear from the context.

The Weyl pseudodistance

Let M be a countable *left-cancellative monoid*, Y a finite set, and $\mathcal{X} = (X_n) \subseteq \mathcal{PF}(M)$ a nondecreasing sequence such that $\bigcup_{n \geqslant 0} X_n = M$.

- The *Weyl pseudodistance* associated to $\mathcal{X}$ is the function $d_{\mathcal{W},\mathcal{X}} : Y^M \times Y^M \to [0,1]$ defined by:

$$d_{\mathcal{W},\mathcal{X}}(f,g) = \limsup_{n\to\infty} \frac{1}{|X_n|} \max_{m \in M} H_{mX_n}(f,g) \text{ for every } f,g : M \to Y$$

  where $mU = \{mx \mid x \in U\}$. Note that $d_{\mathcal{W},\mathcal{X}}(f,g) \geqslant d_{\mathcal{B},\mathcal{X}}(f,g)$
- The *Weyl equivalence* on Y^M associated to $\mathcal{X}$ is defined as:

$$f \sim_{\mathcal{W},\mathcal{X}} g \text{ if and only if } d_{\mathcal{W},\mathcal{X}}(f,g) = 0 \,.$$

- The *Weyl space* associated to $\mathcal{X}$ is the metric space:

$$\mathcal{W}_{\mathcal{X}}(X,Y) = \left(Y^M / \sim_{\mathcal{W},\mathcal{X}}, d_{\mathcal{W},\mathcal{X}} \right)$$

  where $d_{\mathcal{W},\mathcal{X}}(\varphi,\gamma) = d_{\mathcal{W},\mathcal{X}}(f,g)$ for any $f \in \varphi$ and $g \in \gamma$.

We omit M, Y, and/or $\mathcal{X}$ if irrelevant or clear from the context.

The canonical Besicovitch and Weyl spaces

The following spaces, which we dub "canonical", have special relevance in the theory of dynamical systems:

- $\mathcal{B}_0 = \mathcal{B}_{(X_n)}(\mathbb{N}, \{0,1\})$ where $X_n = [0 : n]$;
- $\mathcal{B}_1 = \mathcal{B}_{(X_n)}(\mathbb{Z}, \{0,1\})$ where $X_n = [-n : n]$;
- $\mathcal{W}_0 = \mathcal{W}_{(X_n)}(\mathbb{N}, \{0,1\})$ where $X_n = [0 : n]$;
- $\mathcal{W}_1 = \mathcal{W}_{(X_n)}(\mathbb{Z}, \{0,1\})$ where $X_n = [-n : n]$.

The usual bijection from $\mathbb{N}$ to $\mathbb{Z}$:

$$q(n) = \frac{n}{2} \text{ if } n \text{ is even else } -\frac{n+1}{2}$$

induces two maps $q_{\mathcal{B}} : \mathcal{B}_1 \longrightarrow \mathcal{B}_0$ and $q_{\mathcal{W}} : \mathcal{W}_1 \longrightarrow \mathcal{W}_0$ such that:

- $d_{\mathcal{B}}(b, b') = d_{\mathcal{B}}(q_{\mathcal{B}}(b), q_{\mathcal{B}}(b'))$ for every $b, b' \in \mathcal{B}_1$; and
- $d_{\mathcal{W}}(q_{\mathcal{W}}(w), q_{\mathcal{W}}(w')) \leqslant d_{\mathcal{W}}(w, w') \leqslant 2d_{\mathcal{W}}(q_{\mathcal{W}}(w), q_{\mathcal{W}}(w'))$ for every $w, w' \in \mathcal{W}_1$.

Thus, when we want to prove a topological property of either the Besicovitch or the Weyl space, we can do so indifferently with the "one-sided" version or with the "two-sided" one.

First observations

The Weyl space is *finer-grained* than the Besicovitch space.

- As $d_{\mathcal{W}}(u,v) \geqslant d_{\mathcal{B}}(u,v)$, each class of $\mathcal{B}$-equivalence is a union of classes of $\mathcal{W}$-equivalence.

In general, the Besicovitch space is *strictly* coarser-grained than the Weyl space.

- Let $w = \lambda\, k \,.\, [2^m \leqslant k < 2^m + m]_I \in \{0,1\}^\omega$ where $m = \lfloor \lg k \rfloor \, [k > 0]_I$:

$$w = 00\mathbf{1}0\mathbf{11}00\mathbf{111}00000\mathbf{1111}000000000000\ldots$$

- Then $d_{\mathcal{B}}(w, 0^\omega) = 0$ but $d_{\mathcal{W}}(w, 0^\omega) = 1$.

Each equivalence class has at most one *periodic* representative.

- If $u \neq v$ are periodic of periods p and q, respectively, then $d_{\mathcal{B}}(u,v) \geqslant \dfrac{1}{pq}$.

The canonical spaces are *perfect:* Each point is the limit of the sequence of points *different* from it.

- Given u, let $u_n = \lambda\, i \,.\, u_i \text{ XOR } [i \in n\mathbb{Z}]_I$.
- Then $d_{\mathcal{B}}(u_n, u) = d_{\mathcal{W}}(u_n, u) = \dfrac{1}{n}$ for every $n \geqslant 1$.

Completeness

Theorem (Marcinkiewicz, 1939)

All Besicovitch spaces (in particular, the canonical ones) are complete.

Proof idea for $\mathcal{B}_0$:

- Assume (x_n) is a sequence of infinite words that is Cauchy for $d_{\mathcal{B}}$.
- Take a subsequence that is "Cauchy tightly enough", e.g., $d_{\mathcal{B}}(x_{n_k}, x_{n_{k+1}}) < 2^{-(k+1)}$.
- Use this "tightness" to construct a limit point for $\left([x_n]_{\mathcal{B}}\right)$.

Theorem (Downarowicz and Iwanik, 1988)

$\mathcal{W}_1$ is not complete (nor is $\mathcal{W}_0$).

Proof idea for $\mathcal{W}_1$:

- The number of minimal subsystems of the orbit closure of a point in the shift space is *lower semicontinuous* for the Weyl pseudodistance.
- Use a recursive procedure to construct a sequence (x_n) which converges in the shift space to some x and is Cauchy for the Weyl pseudodistance.
- Reach a contradiction by showing that $\overline{\mathcal{O}(x)}$ must contain both 0^ω and 1^ω.

Sturmian words

Let $\alpha \in (0,1)$ be irrational.

- The *Sturmian word* of *slope* α is the binary infinite word S_α defined by:

$$(S_\alpha)_n = [n\alpha - \lfloor n\alpha \rfloor \geqslant 1 - \alpha]_I \quad \text{for every } n \in \mathbb{N}$$

- For $n \geqslant 1$ this checks if the nth intersection of the straight line of slope α with the square grid is with a *horizontal* line.

Two applications of the *ergodic theorem* give:

Lemma (Blanchard, Formenti, and Kůrka, 1999)

1. For every irrational $\alpha \in (0,1)$,

$$d_{\mathcal{B}}(S_\alpha, 0^\omega) = d_{\mathcal{W}}(S_\alpha, 0^\omega) = \alpha$$

2. For every irrational $\alpha, \beta \in (0,1)$ irrational such that α/β is also irrational,

$$d_{\mathcal{B}}(S_\alpha, S_\beta) = d_{\mathcal{W}}(S_\alpha, S_\beta) = \alpha(1-\beta) + \beta(1-\alpha)$$

Cantor, Besicovitch, Weyl: More differences

Theorem (Blanchard, Formenti, and Kůrka, 1999)

The canonical spaces are neither separable nor locally compact.

Idea of proof for $\mathcal{B}_0$ and $\mathcal{W}_0$:

- For every $r \in (0, 1]$ there exist uncountably many irrationals $\frac{r}{4} < \alpha < \beta < \frac{r}{2}$ such that $\frac{\alpha}{\beta}$ is also irrational.
- Any two such $[S_\alpha]$ and $\left[S_\beta\right]$ are at distance $\frac{r}{4}$ or greater: Hence, any dense subset must be uncountable.
- Every closed disk centered in $[0^\omega]$ contains an infinite subset without limit points: Hence, $[0^\omega]$ does not have a fundamental system of compact neighborhoods.

A substitution procedure

Let $A = \{0,1\} \cup \{\perp\}$. Consider the following substitution procedure on A^ω:

$$s(u,v)_i = \begin{cases} v_k & \text{if } i \text{ is the } k\text{th occurrence of } \perp \text{ in } u\,, \\ u_i & \text{otherwise}\,. \end{cases}$$

For example, $s((0\perp)^\omega, (\perp 1)^\omega) = (0\perp 01)^\omega$.

- Define $f : \{0,1\}^* \to A^\omega$ recursively as follows:

$$f(u) = \begin{cases} \perp^\omega & \text{if } u = \varepsilon\,, \\ s(f(v), (0\perp)^\omega) & \text{if } u = v0\,, \\ s(f(v), (\perp 1)^\omega) & \text{if } u = v1\,. \end{cases}$$

- For $u \in \{0,1\}^\omega$, the sequence $\left(f\left(u_{[0:n]}\right)\right)$ converges in A^ω: we denote its limit by $f(u)$.
- Then, for every $u, v \in \{0,1\}^\omega$, the following are equivalent:
  1. $d_{\mathcal{B}}(f(u), f(v)) = 0$;
  2. $d_{\mathcal{W}}(f(u), f(v)) = 0$;
  3. either $u = v$, or $u = w01^\omega$ and $v = w10^\omega$ for some $w \in \{0,1\}^*$, except possibly for a *unique* symbol $\perp$.

Connectedness and dimensionality

Theorem (Blanchard, Formenti, and Kůrka, 1999)

The canonical spaces are pathwise connected and infinite dimensional.

Proof idea for $\mathcal{W}_0$:

- For $x \in [0,1]$ let $g(x) = f(u)$ where u is the unique binary representation of x which has *infinitely many zeroes*.
- The function $g_{\mathcal{W}} = \lambda\, x : [0,1] \,.\, [g(x)]_{\mathcal{W}} : \mathcal{W}_0$ is continuous and surjective.
- Then for every $w \in \{0,1\}^*$,

  $$\lambda\, x : [0,1] \,.\, [\lambda\, i \,.\, w_i g(x)_i]_{\mathcal{W}}$$

  is a path in $\mathcal{W}_0$ from $[0^\omega]_{\mathcal{W}}$ to $[w]_{\mathcal{W}}$:
  Thus, $\mathcal{W}_0$ is pathwise connected.
- Moreover, for every $d \geqslant 1$, if $h(x)$ is the *interleaving* of the sequences $g(x_1), \ldots, g(x_d)$, then

  $$\lambda\, x : [0,1]^d \,.\, [g(h(x))]_{\mathcal{W}}$$

  is a continuous injective function from $[0,1]^d$ to $\mathcal{W}_0$:
  Thus, $\mathcal{W}_0$ is infinite dimensional.

A comparison of topologies

Cantor	Besicovitch	Weyl
perfect	perfect	perfect
complete	complete	not complete
compact	not locally compact	not locally compact
separable	not separable	not separable
totally disconnected	pathwise connected	pathwise connected
zero-dimensional	infinite-dimensional	infinite-dimensional

Cellular automata

A *cellular automaton* is a tuple $\mathcal{A} = \langle G, A, N, f \rangle$ where:

- G is a countable group;
- A is a finite alphabet;
- the *neighborhood (index)* N is a finite subset of G; and
- $f : A^N \to A$ is the *local rule*.

The synchronous application of the local rule to every cell determines a *global function:*

$$F_{\mathcal{A}}(c) = \lambda\, g : G \,.\, f\left((c \triangleright g)|_N\right) : A \text{ for every } c \in A^G$$

where $c \triangleright g = \lambda\, x \,.\, c(gx)$ is the *translation* by $g \in G$ of $c \in A^G$.

- The shift dynamical system is a CA with $G = \mathbb{Z}$ and $N = \{1\}$.
- The *majority rule* is a CA with $A = \{0, 1\}$, $|N| = m$ odd, and $f(p) = \left[\sum_{i=1}^{m} p_i > m/2\right]_I$.

A characterization of CA global functions

Theorem (Hedlund, 1969)

A function $F : A^G \to A^G$ is the global function of a CA if and only if it is continuous *in the Cantor topology* and satisfies:

$$F(c) \triangleright g = F\left(c \triangleright g\right) \text{ for every } c \in A^G, g \in G$$

- Essentially, a CA is an action in time that commutes with an action in space.
- $\lambda\, c \,.\, c \triangleright g$ is a CA global function if and only if $gh = hg$ for every h.

Cellular automata in the Besicovitch and Weyl spaces

Theorem (Blanchard, Formenti, and Kůrka, 1999)

Let $\mathcal{A} = \langle \mathbb{Z}, A, [-r : r], f \rangle$ be a 1D CA with alphabet A and global function F.

1. The functions $F_{\mathcal{B}} : \mathcal{B}(\mathbb{Z}, A) \to \mathcal{B}(\mathbb{Z}, A)$ and $F_{\mathcal{W}} : \mathcal{W}(\mathbb{Z}, A) \to \mathcal{W}(\mathbb{Z}, A)$ defined by:

$$F_{\mathcal{B}}(\phi) = [f]_{\mathcal{B}} \text{ where } f \in \phi\,; \;\; F_{\mathcal{W}}(\eta) = [h]_{\mathcal{W}} \text{ where } h \in \eta$$

   are continuous and commute with the operators induced by the shift.
2. If one between F, $F_{\mathcal{B}}$, and $F_{\mathcal{W}}$ is surjective, then so are the other two.
3. If F is equicontinuous, then $F_{\mathcal{B}}$ and $F_{\mathcal{W}}$ are equicontinuous.
4. If F is equicontinuous at c, then $F_{\mathcal{B}}$ and $F_{\mathcal{W}}$ are equicontinuous at $\left[(c_{[-m:m]})^{\mathbb{Z}}\right]$, where m is such that if $c_{[-m:m]} = c'_{[-m:m]}$, then $F^n(c)_{[-r:r]} = F^n(c')_{[-r:r]}$ for every n.
5. If one between $F_{\mathcal{B}}$ and $F_{\mathcal{W}}$ is sensitive, then F is sensitive.

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4. If F is equicontinuous at c, then $F_{\mathcal{B}}$ and $F_{\mathcal{W}}$ are equicontinuous at $\left[(c_{[-m:m]})^{\mathbb{Z}}\right]$, where m is such that if $c_{[-m:m]} = c'_{[-m:m]}$, then $F^n(c)_{[-r:r]} = F^n(c')_{[-r:r]}$ for every n.
5. If one between $F_{\mathcal{B}}$ and $F_{\mathcal{W}}$ is sensitive, then F is sensitive.

Furthermore:

The shift is an *isometry* in both $\mathcal{B}$ and $\mathcal{W}$!

The latter is because for every k and m:

$$\begin{aligned} H_{[k:k+n]}(\sigma(c), \sigma(c')) &= H_{[k+1:k+n+1]}(c, c') \\ &= H_{[k:k+n]}(c, c') + O(1) \end{aligned}$$

What could go wrong?

Let $\mathbb{F}_2$ be the *free group on two generators* a, b:

- Elements are *reduced words* over $\{a, b, a^{-1}, b^{-1}\}$, where no pair xx^{-1} or $x^{-1}x$ appears.
- The product of u and v is the *unique* reduced word w that can be obtained by removing all pairs xx^{-1} or $x^{-1}x$ from uv.
- The identity element is the empty word.
- The inverse of $x_1 \dots x_n$ is $x_n^{-1} \cdots x_1^{-1}$.

Let X_n be the set of reduced words of length at most n.

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Let X_n be the set of reduced words of length at most n.

Then $d_{\mathcal{B},\mathcal{X}}$ is not invariant by translations!

- For $g \in \mathbb{F}_2$, $g \neq \varepsilon$, let $i(g)$ be the initial of the unique reduced word that represents g.
- Let $c_0 = \lambda\, c \,.\, 0$ and $c = \lambda\, g \,.\, \left[i(g) = a^{-1}\right]_I$.
- Then $c \triangleright a = \lambda\, g \,.\, [g = \varepsilon \text{ OR } i(g) \neq a]_I$.
- Thus, $d_{\mathcal{B},\mathcal{X}}(c, c_0) = \dfrac{1}{4}$, but $d_{\mathcal{B},\mathcal{X}}(c \triangleright a, c_0 \triangleright a) = \dfrac{3}{4}$.

What makes things right

A countable group G is *amenable* if it satisfies one of the following, equivalent conditions (thus, all of them):

1. For every $K \in \mathcal{PF}(G)$ and every $\varepsilon > 0$ there exists $F \in \mathcal{PF}(G)$ such that $|KF \setminus F| < \varepsilon |F|$.
2. There exist a *left Følner sequence* (F_n) of finite nonempty subsets of G such that:

$$\lim_{n\to\infty} \frac{|gF_n \setminus F_n|}{|F_n|} = 0 \ \text{ for every } g \in G\,.$$

3. There exists a *finitely* additive probability measure μ, defined on *every* subset of G, such that $\mu(gA) = \mu(A)$ for every $g \in G$ and $A \subseteq G$.
4. Any of the above, with multiplication on the right instead of on the left.
5. Any of the above, with *symmetric difference* in place of difference.
6. Any of the above, and the F_n can be taken *symmetric* ($x \in F_n$ if and only if $x^{-1} \in F_n$).

Amenability: Examples and counter-examples

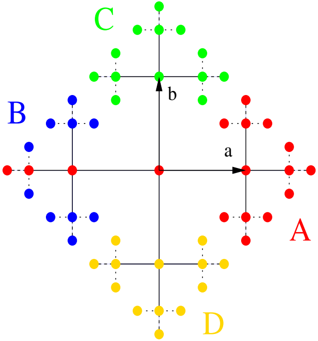
$\mathbb{Z}^d$ is amenable for every $d \geqslant 1$.

- $\left([-n:n]^d\right)$ is a Følner sequence for $\mathbb{Z}^d$, because for every $k \geqslant 1$,

$$\left|[-(n+k):n+k]^d \setminus [-n:n]^d\right| = (2n+2k+1)^2 - (2n+1)^2 = \Theta\left(n^{d-1}\right)$$

$\mathbb{F}_2$ is not amenable.

- Consider the subdivision in figure.
- Then $A \cup B \cup C \cup D = A \cup aB = C \cup bD$.
- So the condition on the probability measure cannot be satisfied.



A new characterization of an old class

Theorem (C., 2009)

Let G be an amenable group and let $\mathcal{X}$ be a Følner sequence for G made of *symmetric* sets.
Let $\mathcal{A}$ be a CA on G with global function F.
If one between F, $F_{\mathcal{B}}$, and $F_{\mathcal{W}}$ is surjective, then so are the other two.

Theorem (C., 2009; C. and Guillon, 2021)

Let G be a countable group and $\mathcal{X} = (X_n)$ an exhaustive sequence for G.

1. The following are equivalent:
   1. The shift is an isometry of $\mathcal{B}_{\mathcal{X}}(G, A)$ for every finite alphabet A.
   2. $\mathcal{X}$ is a left Følner sequence.
2. The following are equivalent:
   1. Every CA on G is continuous in $\mathcal{B}_{\mathcal{X}}(G, A)$ for every finite alphabet A.
   2. $\mathcal{X}$ is a *right* Følner sequence.