An introduction to Sweedler Theory

Estonian-Latvian Theory Days

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Content

Dual and Finite Dual

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sends coalgebras to algebras

$$\rightsquigarrow (-)^* : (\mathrm{Coalg}\, k)^{op} \to \mathrm{Alg}\, k$$

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$$(f \star g)(c) = \Sigma f(c_1) \cdot g(c_2); \Delta c = \Sigma c_1 \otimes c_2$$

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The unit is $C \xrightarrow{\varepsilon} k \xrightarrow{\eta} A$

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This is a Sweedler operation!

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Morally, the finite dual solves the comultiplication issue by ensuring $A^\circ \otimes A^\circ \cong (A \otimes A)^\circ$

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The main example is the cofree coalgebra construction

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The construction of two of the Sweedler operations are based on the (co)free (co)algebra construction.

Measuring Maps

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A map $f : C \otimes A \to B$ is said to be a measuring map, if currying yields an algebra map $A \to [C, B]$. Explicitly,

1. $f(c \otimes a_1 a_2) = \sum f(c_1 \otimes a_1) f(c_2 \otimes a_2)$
2. $f(c \otimes 1_A) = \varepsilon(c) \cdot 1_B$

Example

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Write $\partial^n = f(x^n \otimes -)$, then f is a measuring map if and only if

1. $\partial^n(a_1 a_2) = \sum_{i=0}^{n} \binom{n}{i} \partial^{n-i}(a_1)\partial^i(a_2)$
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$\rightsquigarrow \{\partial^n\}_{n=0}^{\infty}$ is a "differentiation" system.

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ii. $f : A \to B$ induces a measuring map; $\partial^0 = f, \partial^n = 0$ for $n \geq 1$.

Sweedler Operations

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One can show that $- \rhd -$ witnesses $\mathrm{Alg}\, k$ being copowered in $\mathrm{Coalg}\, k$.

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1. $- \triangleright A : \mathrm{Coalg}\, k \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Alg}\, k : \{A, -\}$
2. $\{-, A\} : \mathrm{Alg}\, k \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (\mathrm{Coalg}\, k)^{op} : [-, A]$
3. $C \triangleright - : \mathrm{Alg}\, k \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Alg}\, k : [C, -]$

Graded Categories

For $\mathcal{V}$ a monoidal category.

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For $\mathcal{V}$ a monoidal category.
A $\mathcal{V}$-graded category $\mathcal{C}$ is an essentially an ordinary category where all the morphisms are graded

$$A \xrightarrow{f,V} B$$

Composition:

$$A \xrightarrow{f,V} B \xrightarrow{g,W} C \quad \mapsto \quad A \xrightarrow{\quad gf,W\otimes V \quad} C$$

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$$A \xrightarrow{f,V} B \xrightarrow{g,W} C \;\mapsto\; A \xrightarrow{gf,W\otimes V} C$$

Reindexing:

$$A \xrightarrow{f,V} B \;, \xi : V' \to V \mapsto \; A \xrightarrow{\xi^* f,V'} B$$

Briefly, this is described by a functor

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Thus, we have the following full subcategories:

$$
\begin{tikzcd}
 & \mathcal{V}-\mathit{graded} & \\
\mathcal{V}-\mathit{copowered} \arrow[ur, hook] & \mathcal{V}-\mathit{powered} \arrow[u, hook] & \mathcal{V}-\mathit{enriched} \arrow[ul, hook]
\end{tikzcd}
$$

Upshot: A category being powered / copowered / enriched in a monoidal category, can be seen as properties of graded categories.

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2. $\mathcal{C}$ is copowered iff $(V; A, -)$ is representable
3. $\mathcal{C}$ is powered iff $(V; -, B)$ is corepresentable

The General Framework

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We need a reasonable notion of a V-graded morphism $V; A \to B$ for $A, B \in \mathcal{C}$ and $V \in \mathcal{V}$

The idea is to define what it means for $V \otimes A \to B$ to be measuring, and define
$(V; A, B) = \mathrm{Meas}(V \otimes A, B)$

Note that $\mathcal{W}$ needs some symmetry, since $\mathcal{V}$ must be monoidal.

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$$\begin{tikzcd}
V_1 \otimes V_2 \arrow[rr] \arrow[d, "\Delta\otimes\Delta"'] & & (V_1 \otimes V_2) \otimes (V_1 \otimes V_2) \\
(V_1 \otimes V_1) \otimes (V_2 \otimes V_2) \arrow[urr, dashed, "?"'] & &
\end{tikzcd}$$

Duoidal Categories

A duoidal category $(\mathcal{W}, \otimes, I, *, J)$ consists of structural maps

$$\zeta : (A * B) \otimes (C * D) \to (A \otimes C) * (B \otimes D)$$

$$\Delta : I \to I * I \qquad \mu : J \otimes J \to J \qquad \eta = \varepsilon : I \to J$$

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   We define $f : V \otimes A \to B$ to be measurable whenever $A \to [V, B]$ is a monoid map.
   This makes sense, even if $\otimes$ is not right closed!

Upshot: $\mathcal{C}$ is $\mathcal{V}$-graded, with

$$(V; A, B) = \mathsf{Meas}_*(V \otimes A, B)$$